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Nested potentials and robust equilibria

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Abstract

This paper introduces the notion of nested best-response potentials for complete information games. It is shown that a unique maximizer of such a potential is a Nash equilibrium that is robust to incomplete information in the sense of Kajii and Morris (1997, mimeo).

Keywords: incomplete information, potential games, robustness, refinements.

JEL Classification: C72, C73

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1 Introduction

When analyzing a strategic situation, we often assume the common knowledge of payoffs and describe the situation as a complete information game. However, an equilibrium outcome of a complete information game may be very different from that of an incomplete information game that departs slightly from the complete information game, as demonstrated by Rubinstein (1989) and Carlsson and van Damme (1993). In this light, Kajii and Morris (1997a,b) introduced the concept of equilibria that are robust to incomplete information. A Nash equilibrium of a complete information game is said to be robust to incomplete information if every incomplete information game the payoffs of which differ from those of the original game only very rarely has a Bayesian Nash equilibrium close to the Nash equilibrium.

This paper introduces the concept of nested best-response potentials and provides a new sufficient condition for the robustness of an equilibrium to incomplete information. The nested best-response potentials generalize the best-response potentials introduced by Morris and Ui (2005), applying the idea of ‘nesting’ based on Uno (2007) as follows. A best-response potential of a game is a real-valued function on the set of action profiles of the game that ‘incorporates information’ about every players’ best-response. It is known that a maximizer of a best-response potential is a Nash equilibrium of the game. It is as if the best-response potential is the payoff function of a representative agent that chooses strategies for all players.

In considering a nested best-response potential, we think of a representative agent for a subset T of players, instead of one for all of them: for each player i in T , and for any given belief over strategy profiles of other players, maximizing this representative agent’s payoff f_T yields a best-response for each player i in T . Suppose that there is a partition \mathcal{T} of players such that, for each member T of \mathcal{T} , there is such a representative agent whose payoff function is f_T .³ Then the collection of f_T ’s can be seen as a new complete information game, where each member T in \mathcal{T} is regarded as a single player. That is, the original game is reduced to a game with a smaller number of players.

Notice that such reduction can be nested: the new game among step 1 representative agents may be reduced to a game with an even smaller number of players, by considering a step 2

³This idea also has appeared as q -potential in Monderer (2007).

representative agent for each member of a partition of step 1 representative agents, and then a representative agent for each member of a partition of these, and so on. We say that a game has a nested best-response potential if a game is reduced to a game with one representative agent through this process. We call a unique maximizer of nested best-response potential a nested BRP-maximizer.

The main result of this paper shows that a nested BRP-maximizer is robust to incomplete information in sense of Kajii and Morris (1997b) (Theorem 4.1).

In the literature, various sufficient conditions are given for robustness to incomplete information in sense of Kajii and Morris (1997a,b). Kajii and Morris (1997a) provide sufficient conditions for games with unique correlated equilibria and for games with \mathbf{p} -dominance equilibria with low \mathbf{p} . Ui (2001) provides a sufficient condition for games with exact potential maximizers (P-maximizers) introduced by Monderer and Shapley (1996). Morris and Ui (2005) provide a sufficient condition for games with generalized potential maximizers (GP-maximizers), which strictly generalized the conditions of Kajii and Morris (1997a) and Ui (2001). Morris and Ui (2005) also introduce three special concepts of GP-maximizer: best-response potential maximizers (BRP-maximizers), monotone potential maximizers (MP-maximizers), and local potential maximizers (LP-maximizers).⁴ Tercieux (2006) provides a sufficient condition for games with \mathbf{p} -best-response sets with low \mathbf{p} , which strictly generalized two conditions of Kajii and Morris (1997a,b) but specialized the condition in terms of LP-maximizers, MP-maximizers, and GP-maximizers. Oyama and Tercieux (2009) provide a sufficient condition for games with iterated MP-maximizers, which generalizes the condition in terms of MP-maximizers.⁵ Moreover, Oyama and Tercieux (2009) also introduce two special but tractable concepts of iterated MP-maximizers: iterated LP-maximizers and iterated \mathbf{p} -dominance equilibria, since it is generally a difficult task to find an MP-maximizer and an iterated MP-maximizer.

We show that our condition in terms of nested BRP-maximizers strictly generalizes the conditions in terms of P-maximizers and BRP-maximizers. We also demonstrate that our condition neither implies nor is implied by the conditions in terms of unique correlated equilibria, \mathbf{p} -

⁴Whether the condition in terms of BRP-maximizers implies the condition in terms of MP-maximizers is an open question.

⁵It is not sure whether the condition in terms of iterated MP-maximizers strictly generalizes the condition in terms of MP-maximizers.

dominance equilibria, \mathbf{p} -best-response sets, iterated LP-maximizers and iterated \mathbf{p} -dominance equilibria. However, it is left as an open question whether our condition implies the conditions in terms of GP-maximizers, MP-maximizers, LP-maximizers, and iterated MP-maximizers. We discuss advantages of our condition over the conditions in terms of GP-maximizers, MP-maximizers, LP-maximizers, and iterated MP-maximizers in practical aspects (Remarks 5.11 and 5.18).

2 Robust equilibria

A finite complete information game consists of a finite player set $N = \{1, \dots, n\}$, a finite action set A_i for $i \in N$, and the payoff function $g_i : A \rightarrow \mathbb{R}$ for $i \in N$, where $A := \prod_{i \in N} A_i$. Since we fix the set A of action profiles, we denote a complete information game $(N, (A_i)_{i \in N}, (g_i)_{i \in N})$ simply by $\mathbf{g}^N := (g_i)_{i \in N}$. For notational convenience, we write $a = (a_i)_{i \in N} \in A$; for $i \in N$, $A_{-i} = \prod_{j \neq i} A_j$ and $a_{-i} = (a_j)_{j \neq i} \in A_{-i}$; and for $T \subseteq N$, $A_T = \prod_{i \in T} A_i$, $a_T = (a_i)_{i \in T} \in A_T$, $A_{-T} = \prod_{i \in N \setminus T} A_i$, and $a_{-T} = (a_i)_{i \in N \setminus T} \in A_{-T}$. We write $(a_T, a_{-T}) \in A_T \times A_{-T}$. We write (a_i, a_{-i}) instead of $(a_{\{i\}}, a_{-\{i\}})$ for simplicity. For $i \in N$, a function $f : A \rightarrow \mathbb{R}$ and $X_i \subset A_i$, let denote $BR_i^f(\lambda_i | X_i) := \arg \max_{a_i \in X_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) f(a)$ and $BR_i^f(\lambda_i) := BR_i^f(\lambda_i | A_i)$ for $\lambda_i \in \Delta(A_{-i})$.⁶

Consider an incomplete information game with the player set N and the set A of action profiles. Let Θ_i be a countable set of types of player i . The set of type profiles is $\Theta := \prod_{i \in N} \Theta_i$. We write $\Theta_{-i} = \prod_{j \neq i} \Theta_j$ and $\theta_{-i} = (\theta_j)_{j \neq i} \in \Theta_{-i}$; for $T \subseteq N$, $\Theta_T = \prod_{i \in T} \Theta_i$, $\theta_T = (t_i)_{i \in T} \in \Theta_T$, $\Theta_{-T} = \prod_{i \in N \setminus T} \Theta_i$, and $\theta_{-T} = (\theta_i)_{i \in N \setminus T} \in \Theta_{-T}$. Let $P \in \Delta(\Theta)$ be the common prior probability distribution over the set Θ of type profiles such that for each $i \in N$ and $\theta_i \in \Theta_i$, the marginal probability of θ_i is positive, i.e., $P_i(\theta_i) := \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_i, \theta_{-i}) > 0$. A payoff function for player i is a bounded function $u_i : A \times \Theta \rightarrow \mathbb{R}$. Since we will fix N , Θ , and A throughout the paper, we simply denote an incomplete information game by (P, \mathbf{u}) , where $\mathbf{u} := (u_i)_{i \in N}$.

A strategy of player i is a function $\sigma_i : \Theta_i \rightarrow \Delta(A_i)$. We write Σ_i for the set of strategies of player i , and write $\Sigma = \prod_{i \in N} \Sigma_i$ and $\sigma = (\sigma_i)_{i \in N} \in \Sigma$; $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$ and $\sigma_{-i} = (\sigma_j)_{j \neq i} \in \Sigma_{-i}$;

⁶For a set S , $\Delta(S)$ denotes the set of all probability distributions over S .

for $T \subseteq N$, $\Sigma_T = \prod_{i \in T} \Sigma_i$ and $\sigma_T = (\sigma_i)_{i \in T} \in \Sigma_T$. We write $\sigma_i(a_i|\theta_i)$ for the probability of action a_i given $\sigma_i \in \Sigma_i$ and $\theta_i \in \Theta_i$. For $\sigma \in \Sigma$, we write $\sigma(a|\theta) = \prod_{i \in N} \sigma_i(a_i|\theta_i)$ for $a \in A$ and $\theta \in \Theta$; for $\sigma_{-i} \in \Sigma_{-i}$, $\sigma_{-i}(a_{-i}|\theta_{-i}) = \prod_{j \neq i} \sigma_j(a_j|\theta_j)$ for $a_{-i} \in A_{-i}$ and $\theta_{-i} \in \Theta_{-i}$; for $T \subseteq N$ and $\sigma_T \in \Sigma_T$, $\sigma_T(a_T|\theta_T) = \prod_{i \in T} \sigma_i(a_i|\theta_i)$ for $a_T \in A_T$ and $\theta_T \in \Theta_T$.

A strategy profile $(\sigma_i)_{i \in N} \in \Sigma$ is a (*Bayesian Nash*) *equilibrium* of (P, \mathbf{u}) if, for each $i \in N$, and for each $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) \left[\sum_{a \in A} \sigma(a|\theta_i, \theta_{-i}) u_i(a, (\theta_i, \theta_{-i})) - \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}|\theta_{-i}) u_i((a'_i, a_{-i}), (\theta_i, \theta_{-i})) \right] \geq 0$$

for all $a'_i \in A_i$, where $P(\theta_{-i}|\theta_i) = P(\theta_i, \theta_{-i}) / \sum_{\hat{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \hat{\theta}_{-i})$.

Given a complete information game \mathbf{g}^N and an incomplete information game (P, \mathbf{u}) , for each $i \in N$, consider the subset $\bar{\Theta}_i$ of Θ_i such that, if $\theta_i \in \bar{\Theta}_i$ is realized, i 's payoffs are given by g_i independently of the every types θ_{-i} of the other players:

$$\bar{\Theta}_i = \{\theta_i \in \Theta_i | u_i(a, (\theta_i, \theta_{-i})) = g_i(a) \text{ for all } a \in A, \theta_{-i} \in \Theta_{-i} \text{ with } P(\theta_i, \theta_{-i}) > 0\}.$$

We write $\bar{\Theta} = \prod_{i \in N} \bar{\Theta}_i$. An incomplete information game (P, \mathbf{u}) is a δ -*elaboration* of \mathbf{g}^N if $P(\bar{\Theta}) = 1 - \delta$, where $\delta \in [0, 1]$.

Kajii and Morris (1997a) introduced the robustness of equilibria to all elaborations.

Definition 2.1 An action distribution $\mu \in \Delta(A)$ is *robust to all elaborations* in \mathbf{g}^N if, for any $\varepsilon > 0$, there exists $\bar{\delta} > 0$ such that, for any $0 < \delta \leq \bar{\delta}$, every δ -elaboration of \mathbf{g}^N has an equilibrium σ with $\max_{a \in A} |\mu(a) - \sum_{\theta \in \Theta} P(\theta) \sigma(a|\theta)| \leq \varepsilon$.

Kajii and Morris (1997b) also introduced the following weaker notion of robustness of equilibria to ‘canonical’ elaborations.

A type $\theta_i \in \Theta_i \setminus \bar{\Theta}_i$ is *committed* if player i of this type has a strictly dominant action $a_i^{\theta_i} \in A_i$, i.e., $u_i((a_i^{\theta_i}, a_{-i}), (\theta_i, \theta_{-i})) > u_i((a_i, a_{-i}), (\theta_i, \theta_{-i}))$ for all $a_i \in A_i \setminus \{a_i^{\theta_i}\}$, $a_{-i} \in A_{-i}$, and $\theta_{-i} \in \Theta_{-i}$ with $P(\theta_i, \theta_{-i}) > 0$. A δ -elaboration (P, \mathbf{u}) of \mathbf{g}^N is *canonical* if, for each $i \in N$, every $\theta_i \in \Theta_i \setminus \bar{\Theta}_i$ is a committed type.

Definition 2.2 An action distribution $\mu \in \Delta(A)$ is *robust to canonical elaborations* in \mathbf{g}^N if, for every $\varepsilon > 0$, there exists $\bar{\delta} > 0$ such that, for all $0 < \delta \leq \bar{\delta}$, any canonical δ -elaboration of \mathbf{g}^N has an equilibrium σ with $\max_{a \in A} |\mu(a) - \sum_{\theta \in \Theta} P(\theta) \sigma(a|\theta)| \leq \varepsilon$.

It is clear that if an action distribution is robust to all elaborations, then it is also robust to canonical elaborations.⁷

3 Nested potentials

This section introduces the notion of nested best-response potential for complete information games. The nested best-response potentials generalize the best-response potentials defined by Morris and Ui (2005). A best-response potential of a complete information game \mathbf{g}^N is a real valued function f on the set A of action profiles such that, for each player i and any i 's belief $\lambda_i \in \Delta(A_{-i})$ over the set A_{-i} of other players' actions, i 's best-response against the belief λ_i in the alternative game where i 's payoff function equals f , is also his best-response in the original game \mathbf{g}^N .⁸

Definition 3.1 (Morris and Ui, 2005) A function $f : A \rightarrow \mathbb{R}$ is a *best-response potential* of \mathbf{g}^N if, for each $i \in N$, $BR_i^f(\lambda_i) \subseteq BR_i^{g_i}(\lambda_i)$ for all $\lambda_i \in \Delta(A_{-i})$. An action profile a^* is a *BRP-maximizer* if $\{a^*\} = \arg \max_{a \in A} f(a)$.

We generalize the best-response potentials by means of the ‘nested construction’ proposed in Uno (2007) as follows: firstly, for a partition \mathcal{T} of N , we define the \mathcal{T} -best-response potentials:⁹

Definition 3.2 Let \mathcal{T} be a partition of N . A *best-response \mathcal{T} -potential* of \mathbf{g}^N is a tuple $(\mathcal{T}, (A_T)_{T \in \mathcal{T}}, (f_T)_{T \in \mathcal{T}})$, where, for each $T \in \mathcal{T}$, $f_T : A \rightarrow \mathbb{R}$ satisfies that, for each $i \in T$,

⁷Whether or not the converse holds is an open question.

⁸There are three versions of best-response potential in the literature. The best-response potential of Morris and Ui (2005) is a cardinal version of the pseudo-potentials introduced in Dubey *et al.* (2006). The one of Morris and Ui (2004) is a version of best-response potential where the inclusion in the definition is replaced by the equality. The one of Voorneveld (2000) is an ordinal version of best-response potential of Morris and Ui (2004).

⁹The idea of games with partition \mathcal{T} -potentials is same as that of q -potential games defined by Monderer (2007) independently and earlier than Uno (2007): a game \mathbf{g}^N is a q -potential game if and only if \mathbf{g}^N has a partition \mathcal{T} -potential, where q refers to the number of elements in \mathcal{T} . For convenience to define nested potentials we use the partition \mathcal{T} -potentials.

$BR_i^{f_T}(\lambda_i) \subseteq BR_i^{g_i}(\lambda_i)$ for all $\lambda_i \in \Delta(A_{-i})$.

We denote such a \mathcal{T} -best-response potential $(\mathcal{T}, (A_T)_{T \in \mathcal{T}}, (f_T)_{T \in \mathcal{T}})$ by $\mathbf{f}^{\mathcal{T}} := (f_T)_{T \in \mathcal{T}}$ since action sets $(A_T)_{T \in \mathcal{T}}$ can be derived from the partition \mathcal{T} of N and the set A of action profiles in the original game \mathbf{g}^N . Note that any game has a best-response \mathcal{T} -potential for the finest partition $\mathcal{T} = \{\{i\} | i \in N\}$. Note also that a game with a best-response potential is equivalent to a game with a best-response \mathcal{T} -potential for the coarsest partition $\mathcal{T} = \{N\}$.

Notice that we can regard each \mathcal{T} -best-response potential $\mathbf{f}^{\mathcal{T}}$ as a strategic form game, where \mathcal{T} is the player set; for each $T \in \mathcal{T}$, A_T is the action set of T ; and for each $T \in \mathcal{T}$, f_T is the payoff function of T . The idea underlying the notion of the nested best-response potentials is to construct such games iteratively:

Definition 3.3 A function $f : A \rightarrow \mathbb{R}$ is a *nested best-response potential* of \mathbf{g}^N if there exist a finite sequence $\{\mathcal{T}^k\}_{k=0}^K$ of partitions of N and a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K = ((f_T^k)_{T \in \mathcal{T}^k})_{k=0}^K$ of tuples such that

- $\{\mathcal{T}^k\}_{k=0}^K$ is a nested sequence of partitions of N : $\{\mathcal{T}^k\}_{k=0}^K$ is an increasingly coarser sequence of partitions of N with $\mathcal{T}^0 = \{\{i\} | i \in N\}$ and $\mathcal{T}^K = \{N\}$;
- $\mathbf{f}^{\mathcal{T}^0} = (f_T^0)_{T \in \mathcal{T}^0}$ is the original game \mathbf{g}^N : for each $i \in N$, $f_{\{i\}}^0(a) = g_i(a)$ for all $a \in A$;
- for each $k = 1, 2, \dots, K$, $\mathbf{f}^{\mathcal{T}^k} = (f_T^k)_{T \in \mathcal{T}^k}$ is a \mathcal{T}^k -best-response potential of $\mathbf{f}^{\mathcal{T}^{k-1}} = (f_T^{k-1})_{T \in \mathcal{T}^{k-1}}$, where $\mathbf{f}^{\mathcal{T}^{k-1}}$ is regarded as a strategic form game as above: for each $T^k \in \mathcal{T}^k$ and for each $T^{k-1} \in \mathcal{T}^{k-1}$ with $T^{k-1} \subseteq T^k$,

$$\begin{aligned} & \arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} \sum_{a_{-T^{k-1}} \in A_{-T^{k-1}}} \lambda_{T^{k-1}}(a_{-T^{k-1}}) f_{T^k}^k(a_{T^{k-1}}, a_{-T^{k-1}}) \\ & \subseteq \arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} \sum_{a_{-T^{k-1}} \in A_{-T^{k-1}}} \lambda_{T^{k-1}}(a_{-T^{k-1}}) f_{T^{k-1}}^{k-1}(a_{T^{k-1}}, a_{-T^{k-1}}) \end{aligned} \quad (1)$$

for all $\lambda_{T^{k-1}} \in \Delta(A_{-T^{k-1}})$; and

- $\mathbf{f}^{\mathcal{T}^K} = (f_N^K)$ is such that $f_N^K(a) = f(a)$ for all $a \in A$.

An action profile a^* is a *nested BRP-maximizer* if $\{a^*\} = \arg \max_{a \in A} f(a)$.

It is clear that if a game has a BRP-maximizer, then it has a nested BRP-maximizer. Nevertheless, even if a game has a nested BRP-maximizer, it may not have a BRP-maximizer as shown later (Example 5.1).

4 Nested potentials and robust equilibria

This section provides a new sufficient condition for the robustness of equilibria in terms of nested BRP-maximizers.

Theorem 4.1 *If \mathbf{g}^N has a nested BRP-maximizer a^* , then the action distribution $\mu \in \Delta(A)$ such that $\mu(a^*) = 1$ is robust to canonical elaborations in \mathbf{g}^N .*

We can show this theorem by arguments similar to those of Theorem 6 in Morris and Ui (2005). Indeed, we replace Lemma 6 of Morris and Ui (2005) by Lemma 4.3 below. Let (P, \mathbf{u}) be a canonical δ -elaboration of \mathbf{g}^N and consider the set of i 's strategies of (P, \mathbf{u}) such that each committed type $\theta_i \in \Theta_i \setminus \bar{\Theta}_i$ chooses the strictly dominant action $a_i^{\theta_i}$.¹⁰

$$\Xi_i := \{\xi_i : \Theta_i \rightarrow A_i \mid \xi_i(\theta_i) = a_i^{\theta_i} \text{ for } \theta_i \in \Theta_i \setminus \bar{\Theta}_i\}.$$

Let $\Xi := \prod_{i \in N} \Xi_i = \{\xi : \Theta \rightarrow A \mid \xi(\theta) = (\xi_i(\theta_i))_{i \in N} \text{ for all } \theta \in \Theta, \text{ and } \xi_i \in \Xi_i \text{ for all } i \in N\}$. For $T \subseteq N$, $\Xi_T := \prod_{i \in T} \Xi_i$.

Note that if (P, \mathbf{u}) is canonical then Ξ is nonempty (Morris and Ui, 2005, Lemma 4).

Let (P, \mathbf{u}) be a canonical δ -elaboration of a complete information game \mathbf{g}^N with a nested best-response potential $f : A \rightarrow \mathbb{R}$. Define a function $V : \Xi \rightarrow \mathbb{R}$ such that

$$V(\xi) := \sum_{\theta \in \Theta} P(\theta) f(\xi(\theta))$$

for all $\xi \in \Xi$ and consider the set of its maximizers $\Xi^* := \arg \max_{\xi \in \Xi} V(\xi)$.

¹⁰Indeed, we set a domain \mathcal{A} of generalized potential of Morris and Ui (2005) to $\prod_{i \in N} \{\{a_i\} \mid a_i \in A_i\}$.

The function V is constructed in a similar way to that of generalized potentials in Morris and Ui (2005). We can show the following lemma by an argument similar to Lemma 5 in Morris and Ui (2005).

Lemma 4.2 *If Ξ is nonempty then Ξ^* is nonempty. If $\xi^* \in \Xi^*$ then*

$$\sum_{\theta \in \Theta, \xi^*(\theta) = a^*} P(\theta) \geq 1 - \delta\kappa,$$

where κ is a positive constant.¹¹

We show that there exists an equilibrium of (P, \mathbf{u}) assigning probability 1 to a maximizer $\xi^* \in \Xi^*$ of V , which corresponds to Lemma 6 in Morris and Ui (2005).

Lemma 4.3 *Suppose \mathbf{g}^N has a nested best-response potential f and (P, \mathbf{u}) is a canonical δ -elaboration of \mathbf{g}^N . For $\xi^* \in \Xi^*$, (P, \mathbf{u}) has an equilibrium $\sigma^* \in \Sigma$ such that $\sigma^*(\cdot|\theta)$ assigns probability 1 to the action $\xi^*(\theta)$ for all $\theta \in \Theta$, i.e., $\sigma^*(\xi^*(\theta)|\theta) = 1$ for all $\theta \in \Theta$.*

Proof. Let $\xi^* \in \Xi^*$. We want to show that, for each $i \in N$ and for each $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) [u_i(\xi^*(\theta), (\theta_i, \theta_{-i})) - u_i((a_i, \xi_{-i}^*(\theta_{-i})), (\theta_i, \theta_{-i}))] \geq 0 \quad (2)$$

for all $a_i \in A_i$. Fix any $i \in N$ and $\theta_i \in \Theta_i$. If $\theta_i \in \Theta_i \setminus \bar{\Theta}_i$, then (2) is true, since $\xi_i^*(\theta_i)$ is the strictly dominant action $a_i^{\theta_i}$ of θ_i .

Suppose that $\theta_i \in \bar{\Theta}_i$. Let a positive integer K and sequences $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K$ and $(T^k)_{k=0}^K$ be such that, for each $k = 0, 1, \dots, K$, $\{i\} = T^0 \subseteq T^1 \subseteq \dots \subseteq T^{K-1} \subseteq T^K = N$, $T^k \in \mathcal{T}^k$ and, $f = f_{T^K}^K$, so that f is a nested best-response potential.

Firstly, since $\xi^* \in \arg \max_{\xi \in \Xi} \sum_{\theta \in \Theta} P(\theta) f(\xi(\theta)) = \arg \max_{\xi \in \Xi} \sum_{\theta \in \Theta} P(\theta) f_{T^K}(\xi(\theta))$, we have

$$\sum_{\theta \in \Theta} P(\theta) [f_{T^K}^K(\xi^*(\theta)) - f_{T^K}^K(\xi_{T^{K-1}}(\theta_{T^{K-1}}), \xi_{-T^{K-1}}^*(\theta_{-T^{K-1}}))] \geq 0$$

¹¹Or, $\kappa > 0$ is independent to δ . For example, $\kappa = [f(a^*) - \min_{a \in A} f(a)]/[f(a^*) - \max_{a \in A \setminus \{a^*\}} f(a)]$.

for all $\xi_{T^{K-1}} \in \Xi_{T^{K-1}}$. It is equivalent to, for each $\theta_{T^{K-1} \setminus \{i\}} \in \Theta_{T^{K-1} \setminus \{i\}}$ with $P_{T^{K-1}}(\theta_i, \theta_{T^{K-1} \setminus \{i\}}) > 0$,

$$\sum_{\theta_{-T^{K-1}} \in \Theta_{-T^{K-1}}} P(\theta_{-T^{K-1}} | \theta_{T^{K-1}}) [f_{T^K}^K(\xi^*(\theta)) - f_{T^K}^K(a_{T^{K-1}}, \xi_{-T^{K-1}}^*(\theta_{-T^{K-1}}))] \geq 0$$

for all $a_{T^{K-1}} \in A_{T^{K-1}}$. Since $\mathbf{f}^{\mathcal{T}^K}$ is a \mathcal{T}^K -best-response potential of $\mathbf{f}^{\mathcal{T}^{K-1}}$, by (1), for each $\theta_{T^{K-1} \setminus \{i\}} \in \Theta_{T^{K-1} \setminus \{i\}}$ with $P_{T^{K-1}}(\theta_i, \theta_{T^{K-1} \setminus \{i\}}) > 0$, we have

$$\sum_{\theta_{-T^{K-1}} \in \Theta_{-T^{K-1}}} P(\theta_{-T^{K-1}} | \theta_{T^{K-1}}) [f_{T^{K-1}}^{K-1}(\xi^*(\theta)) - f_{T^{K-1}}^{K-1}(a_{T^{K-1}}, \xi_{-T^{K-1}}^*(\theta_{-T^{K-1}}))] \geq 0 \quad (3)$$

for all $a_{T^{K-1}} \in A_{T^{K-1}}$.

Next, (3) is equivalent to

$$\sum_{\theta \in \Theta} P(\theta) [f_{T^{K-1}}^{K-1}(\xi^*(\theta)) - f_{T^{K-1}}^{K-1}(\xi_{T^{K-1}}(\theta_{T^{K-1}}), \xi_{-T^{K-1}}^*(\theta_{-T^{K-1}}))] \geq 0 \quad (4)$$

for all $\xi_{T^{K-1}} \in \Xi_{T^{K-1}}$. Since $T^{K-2} \subseteq T^{K-1}$, we have $T^{K-1} = T^{K-2} \cup T^{K-1} \setminus T^{K-2}$, and so $(\xi_{T^{K-2}}, \xi_{T^{K-1} \setminus T^{K-2}}^*) \in \Xi_{T^{K-1}}$ for all $\xi_{T^{K-2}} \in \Xi_{T^{K-2}}$. Thus, (4) implies that

$$\sum_{\theta \in \Theta} P(\theta) [f_{T^{K-1}}^{K-1}(\xi^*(\theta)) - f_{T^{K-1}}^{K-1}(\xi_{T^{K-2}}(\theta_{T^{K-2}}), \xi_{-T^{K-2}}^*(\theta_{-T^{K-2}}))] \geq 0$$

for all $\xi_{T^{K-2}} \in \Xi_{T^{K-2}}$, where $\xi_{-T^{K-2}}^*(\theta_{-T^{K-2}}) = (\xi_{T^{K-1} \setminus T^{K-2}}^*(\theta_{T^{K-1} \setminus T^{K-2}}), \xi_{-T^{K-1}}^*(\theta_{-T^{K-1}}))$ for all $\theta_{-T^{K-2}} \in \Theta_{-T^{K-2}}$. By arguments similar to those given above, we have, for each $\theta_{T^{K-2} \setminus \{i\}} \in \Theta_{T^{K-2} \setminus \{i\}}$ with $P_{T^{K-2}}(\theta_i, \theta_{T^{K-2} \setminus \{i\}}) > 0$,

$$\sum_{\theta_{-T^{K-2}} \in \Theta_{-T^{K-2}}} P(\theta_{-T^{K-2}} | \theta_{T^{K-2}}) [f_{T^{K-2}}^{K-2}(\xi^*(\theta)) - f_{T^{K-2}}^{K-2}(a_{T^{K-2}}, \xi_{-T^{K-2}}^*(\theta_{-T^{K-2}}))] \geq 0$$

for all $a_{T^{K-2}} \in A_{T^{K-2}}$.

By applying the arguments above to $K - 3, K - 4, \dots, 0$, iteratively, we have

$$\sum_{\theta_{-T^0} \in \Theta_{-T^0}} P(\theta_{-T^0} | \theta_{T^0}) [f_{T^0}^0(\xi^*(\theta)) - f_{T^0}^0(a_{T^0}, \xi_{-T^0}^*(\theta_{-T^0}))] \geq 0$$

for all $a_{T^0} \in A_{T^0}$. Since $T^0 = \{i\}$ and $f_{T^0} = g_i$, we have

$$\sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i} | \theta_i) [g_i(\xi^*(\theta)) - g_i(\xi_i(\theta_i), \xi_{-i}^*(\theta_{-i}))] \geq 0$$

for all $a_i \in A_i$. Since $\theta_i \in \bar{\Theta}_i$, we have (2). ■

Lemma 4.2 and 4.3 imply that (P, \mathbf{u}) has an equilibrium $\sigma^* \in \Sigma$ such that $\sigma(\xi^*(\theta) | \theta) = 1$ for all $\theta \in \Theta$, where $\xi^* \in \Xi^*$, and

$$\begin{aligned} \sum_{\theta \in \Theta} P(\theta) \sigma^*(a^* | \theta) &\geq \sum_{\theta \in \Theta, \xi^*(\theta) = a^*} P(\theta) \sigma^*(a^* | \theta) \\ &= \sum_{\theta \in \Theta, \xi^*(\theta) = a^*} P(\theta) \geq 1 - \delta\kappa. \end{aligned}$$

Thus, for each $\varepsilon > 0$, if we choose $\bar{\delta} = \varepsilon/\kappa > 0$, then, for each $\delta \leq \bar{\delta}$, every canonical δ -elaboration (P, \mathbf{u}) of \mathbf{g}^N has an equilibrium σ^* such that $1 - \sum_{\theta \in \Theta} P(\theta) \sigma^*(a^* | \theta) \leq \varepsilon$, which completes the proof.

5 Related literature

The remaining of the paper shows the relationships between our condition (Theorem 4.1) and the other sufficient conditions in the literature.

5.1 BRP-maximizer versus nested BRP-maximizer

Morris and Ui (2005) shows that a BRP-maximizer is robust to canonical elaborations. Our condition strictly generalizes the condition in terms of BRP-maximizers as shown in the following example.

T	L	C	R	B_1	L	C	R	B_2	L	C	R
U	5, 5, 5	0, 0, 0	3, 3, 0	U	4, 4, 4	0, 0, 0	2, 2, 2	U	4, 4, 4	0, 0, 2	3, 3, 0
M	0, 0, 0	0, 0, 2	4, 4, 0	M	2, 2, 2	3, 3, 0	0, 0, 2	M	0, 0, 0	0, 0, 0	2, 2, 0
D	3, 3, 0	4, 4, 0	0, 0, 0	D	0, 0, 0	0, 0, 2	3, 3, 0	D	3, 3, 2	2, 2, 0	0, 0, 2

Table 1: A game (g_1, g_2, g_3)

	U, L	U, C	U, R	M, L	M, C	M, R	D, L	D, C	D, R
T	5, 5	0, 0	0, 3	0, 0	2, 0	0, 4	0, 3	0, 4	0, 0
B_1	4, 4	0, 0	2, 2	2, 2	0, 3	2, 0	0, 0	2, 0	0, 3
B_2	4, 4	2, 0	0, 3	0, 0	0, 0	0, 2	2, 3	0, 2	2, 0

Table 2: A partition $\{\{3\}, \{1, 2\}\}$ -potential $(f_{\{3\}}^1, f_{\{1,2\}}^1)$

Example 5.1 Consider the three-person game $\mathbf{g}^{\{1,2,3\}} = (g_1, g_2, g_3)$ represented in Table 1, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.¹² The game $\mathbf{g}^{\{1,2,3\}}$ has no BRP-maximizer. Indeed, note that $\mathbf{g}^{\{1,2,3\}}$ has a strict best-response cycle $(M, C, T) \rightarrow (M, R, T) \rightarrow (M, R, B_1) \rightarrow (M, C, B_1) \rightarrow (M, C, T)$. Since games with a pseudo-potential cannot have strict best-response cycles as shown by Kukushkin (2004), then games with a best-response potential, which is a special form of pseudo-potentials, cannot have either. Thus $\mathbf{g}^{\{1,2,3\}}$ has no BRP-maximizer.

Nevertheless, the game $\mathbf{g}^{\{1,2,3\}}$ has a nested BRP-maximizer (U, L, T) . Indeed, $(f_{\{3\}}^1, f_{\{1,2\}}^1)$ represented in Table 2 is a $\{\{3\}, \{1, 2\}\}$ -best-response potential of $\mathbf{g}^{\{1,2,3\}}$, where $f_{\{3\}}^1(\cdot) = g_3(\cdot)$ and $f_{\{1,2\}}^1(\cdot) = g_1(\cdot) = g_2(\cdot)$, and considering the $\{\{3\}, \{1, 2\}\}$ -best-response potential $(f_{\{3\}}^1, f_{\{1,2\}}^1)$ as a two-person game, we can show that $\mathbf{f}^{\{1,2,3\}} = (f)$ represented in Table 3 is a $\{\{1, 2, 3\}\}$ -best-response potential of $(f_{\{3\}}^1, f_{\{1,2\}}^1)$. Thus $\mathbf{g}^{\{1,2,3\}}$ has a nested best-response potential f and (U, L, T) is a nested BRP-maximizer.

Remark 5.2 In fact, Morris and Ui (2005) define a more general form of best-response potentials, a \mathcal{P} -measurable best-response potential for a partition \mathcal{P} of the set A , and provide a sufficient condition in terms of \mathcal{P} -measurable BRP-maximizers. If \mathcal{P} is the finest partition, i.e., $\mathcal{P} = \prod_{i \in N} \{\{a_i\} | a_i \in A_i\}$, then a \mathcal{P} -measurable best-response potential is given by Definition

¹² $\mathbf{g}^{\{1,2,3\}}|_T$ restricted by T has a payoff structure similar to the game in Ui (2001, p.1376).

	U, L	U, C	U, R	M, L	M, C	M, R	D, L	D, C	D, R
T	5	0	0	0	2	0	0	0	0
B_1	4	0	2	2	0	2	0	2	0
B_2	4	2	0	0	0	0	2	0	2

Table 3: A nested potential f

3.1. We can show that, for any partition \mathcal{P} , (U, C, L) is not a \mathcal{P} -measurable BRP-maximizer. See Appendix A.1.

Remark 5.3 Ui (2001) shows that an exact potential maximizer (P-maximizers) defined by Monderer and Shapley (1996) is robust to canonical elaborations. By Example 5.1, our condition strictly generalizes the condition in terms of P-maximizers, since the best-response potentials strictly generalize the exact potentials.¹³

5.2 Iterated MP-maximizer versus nested BRP-maximizer

Oyama and Tercieux (2009) introduce the iterated MP-maximizer and provide a sufficient condition for the robustness of equilibria in terms of iterated MP-maximizers.

For $i \in N$, let $A_i = \{0, \dots, m_i\}$.¹⁴ For $i \in N$, we endow $\Delta(A_{-i})$ with the sup norm: $|\lambda_i| = \max_{a_{-i} \in A_{-i}} \lambda_i(a_{-i})$ for $\lambda_i \in \Delta(A_{-i})$. For $\varepsilon > 0$, denote $B_\varepsilon(\lambda_i) = \{\lambda'_i \in \Delta(A_{-i}) \mid |\lambda'_i - \lambda_i| < \varepsilon\}$ and write $B_\varepsilon(\Delta(A_{-i})) = \prod_{\lambda_i \in \Delta(A_{-i})} B_\varepsilon(\lambda_i)$.

Definition 5.4 (Morris and Ui, 2005; Oyama and Tercieux, 2009) Let X^* and X be intervals such that $X^* \subset X \subset A$. X^* is an *MP-maximizer set of \mathbf{g}^N relative to X* if there exist a function $v : A \rightarrow \mathbb{R}$ and a real number $\varepsilon > 0$ such that $X^* = \arg \max_{a \in A} v(a)$, and for each $i \in N$ and all $\lambda_i \in B_\varepsilon(\Delta(X_{-i}))$,

$$\begin{aligned} \min BR_i^v(\lambda_i | [\min X_i, \min X_i^*]) &\leq \min BR_i^{g_i}(\lambda_i | [\min X_i, \max X_i^*]), \text{ and} \\ \max BR_i^v(\lambda_i | [\max X_i^*, \max X_i]) &\geq \max BR_i^{g_i}(\lambda_i | [\min X_i^*, \max X_i]). \end{aligned}$$

¹³See Morris and Ui (2004).

¹⁴In fact, we can consider a more general case such that A_i is a linearly ordered set for $i \in N$.

Definition 5.5 (Oyama and Tercieux, 2009) An action profile $a^* \in A$ is an *iterated MP-maximizer* of \mathbf{g}^N if there exists a sequence of intervals $A = X^0 \supset X^1 \supset \dots \supset X^K = \{a^*\}$ such that X^k is an MP-maximizer set of \mathbf{g}^N relative to X^{k-1} for each $k = 1, \dots, K$.

A game \mathbf{g}^N is said to be *supermodular* for $i \in N$ if, $a_i, a'_i \in A_i$ with $a_i < a'_i$ and for $a_{-i}, a'_{-i} \in A_{-i}$ with $a_{-i} \leq a'_{-i}$, $g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}) \leq g_i(a_i, a'_{-i}) - g_i(a'_i, a'_{-i})$.

Theorem 5.6 (Oyama and Tercieux, 2009) Suppose that \mathbf{g}^N has an iterated MP-maximizer a^* with associated intervals $(X^k)_{k=0}^K$ and monotone potentials $(v^k)_{k=0}^K$. If, for each $k = 0, \dots, K$, $\mathbf{g}^N|_{X_i^{k-1} \times A_{-i}}$ or $v^k|_{X_i^{k-1} \times A_{-i}}$ is supermodular for each $i \in N$ then a^* is robust to all elaborations in \mathbf{g}^N .

We provide a necessary condition for MP-maximizer sets.¹⁵

Lemma 5.7 If an interval $X^* \subseteq A$ is an MP-maximizer set of \mathbf{g}^N relative to an interval $X \subseteq A$ then, for each $i \in N$, for each $\lambda_i \in \Delta(X_{-i}^*)$, $BR_i^{g_i}(\lambda_i|X_i) \cap X_i^* \neq \emptyset$.

Proof. Suppose that \mathbf{g}^N has an MP-maximizer set X^* relative to X with an associated monotone potential v but there exist $i \in N$ and $\lambda_i \in \Delta(X_{-i}^*)$ such that $BR_i^{g_i}(\lambda_i|X_i) \cap X_i^* = \emptyset$. Assume $\max BR_i^{g_i}(\lambda_i|X_i) < \min X_i^*$. Then $\min BR_i^{g_i}(\lambda_i|[\min A_i, \max X_i^*]) = \min BR_i^{g_i}(\lambda_i|X_i)$. Since v is a monotone potential, we have $\min BR_i^v(\lambda_i|[\min A_i, \min X_i^*]) \leq \min BR_i^{g_i}(\lambda_i|[\min A_i, \max X_i^*])$. Since $\min BR_i^v(\lambda_i|[\min A_i, \min X_i^*]) \leq \min BR_i^{g_i}(\lambda_i|[\min A_i, \max X_i^*]) = \min BR_i^{g_i}(\lambda_i|X_i) \leq \max BR_i^{g_i}(\lambda_i|X_i) < \min X_i^*$, we have $\min BR_i^v(\lambda_i|[\min A_i, \min X_i^*]) < \min X_i^*$. Since X^* is an MP-maximizer, $\min BR_i^v(\lambda_i|[\min A_i, \min X_i^*]) = \min X_i^*$, a contradiction. By the similar arguments, if $\max BR_i^{g_i}(\lambda_i|X_i) > \max X_i^*$, we also have a contradiction. ■

For a fixed order on A , even if our condition applies to a game, Theorem 5.6 may not apply to the game, as shown in Example 5.8.

Example 5.8 Consider the game $\mathbf{g}^{\{1,2,3\}}$ represented as in Table 1 again. Assume $\mathbf{g}^{\{1,2,3\}}$ has ordered action sets such that $U < M < D$, $L < C < R$, and $T < B_1 < B_2$. Theorem 5.6 does not apply to $\mathbf{g}^{\{1,2,3\}}$. Indeed, note that $\mathbf{g}^{\{1,2,3\}}$ is not supermodular for $i \in N$. We will show that A is

¹⁵I am grateful to Olivier Tercieux for his suggestion. Discussion in Examples 5.8 and 5.17 was more complicated before his suggestion.

a unique iterated MP-maximizer such that an associated monotone potential is supermodular. By Lemma 5.7, it is easy to show that only A , $\{U\} \times \{L\} \times \{T\}$, $\{U\} \times \{L\} \times \{T, B_1\}$, $\{U\} \times \{L\} \times \{T, B_2\}$, or $\{U\} \times \{L\} \times \{T, B_1, B_2\}$ may be MP-maximizer sets relative to A . Fix any $X_3 \in \{\{T\}, \{T, B_1\}, \{T, B_2\}, \{T, B_1, B_2\}\}$. Now, suppose that there exists a supermodular monotone-potential v with MP-maximizer $\{U\} \times \{L\} \times X_3$. Let $\lambda_1 \in \Delta(A_{-1})$ be such that $\lambda_1(L, T) = \lambda_1(C, T) = 1/2$. Then we have $\{D\} = BR_1^{g_1}(\lambda_1)$. Since v is a monotone-potential, we have $\max BR_1^v(\lambda_1) = \{D\}$. That is, we have

$$v(D, L, T) + v(D, C, T) \geq v(U, L, T) + v(U, C, T). \quad (5)$$

Similarly, let $\lambda_2 \in \Delta(A_{-2})$ be such that $\lambda_2(U, T) = \lambda_2(M, T) = 1/2$. Then we have $\{R\} = BR_2^{g_2}(\lambda_2)$. Since v is a monotone-potential, we have $\max BR_2^v(\lambda_2) = \{R\}$. That is, we have

$$v(U, R, T) + v(M, R, T) \geq v(U, L, T) + v(M, L, T). \quad (6)$$

Since v is supermodular, we have

$$v(U, C, T) - v(U, R, T) \geq v(D, C, T) - v(D, R, T), \quad (7)$$

$$v(M, L, T) - v(D, L, T) \geq v(M, R, T) - v(D, R, T). \quad (8)$$

By summing up inequalities (5)-(8), we have $2v(D, R, T) \geq 2v(U, L, T)$, which contradicts to the assumption $\{U\} \times \{L\} \times X_3$ is an MP-maximizer. Thus, the game has a unique MP-maximizer set A relative to A . Hence we does not apply Theorem 5.6 to the game under the ordered action sets.

Remark 5.9 It is not sure whether or not our condition, as well as that in terms of BRP-maximizers, implies the conditions in terms of iterated MP-maximizers and MP-maximizers. In fact, consider the game $\mathbf{g}^{\{1,2,3\}}$ represented as in Table 1 again. If $\mathbf{g}^{\{1,2,3\}}$ has other ordered action sets such that $U < D < M$, $L < R < C$ and $T < B_1 < B_2$, we can find an MP-maximizer (U, L, T) such that an associate monotone potential is supermodular. Such a monotone potential v is given by Table 4. Thus Theorem 5.6 applies to the game.

T	L	R	C	B_1	L	R	C	B_2	L	R	C
U	50	4	-195	U	40	23	-175	U	40	24	-174
D	-50	4	5	D	-30	24	25	D	-29	25	26
M	-65	27	28	M	-45	47	48	M	-44	48	49

Table 4: A supermodular monotone potential v

Remark 5.10 Morris and Ui (2005) introduce GP-maximizers and provide a sufficient condition in terms of GP-maximizers. The condition generalizes the condition in terms of MP-maximizers. So, it is also not sure whether our condition does not imply the condition in terms of GP-maximizers, as well as that in terms of MP-maximizers.

Remark 5.11 Our condition has advantages over the conditions in terms of (iterated) MP-maximizers and GP-maximizers in practical aspects. Generally, finding an (iterated) MP-maximizer or a GP-maximizer is a hard task because no simple characterization for monotone potentials or generalized potentials is known. If a game is not supermodular, it becomes a harder task to apply the condition in terms of (iterated) MP-maximizers because we need to find an (iterated) MP-maximizer such that an associated (iterated) monotone potential is supermodular. Moreover, whether an (iterated) MP-maximizer exists or not depends on an order over the set A of action profiles, which is shown in Example 5.8 and Remark 5.9; and whether a GP-maximizer exists or not also depends on a covering over the set A of action profiles, which is a domain of generalized potentials. However, how to choose an order for existence of an (iterated) MP-maximizer and a partition for existence of a GP-maximizer are unknown.

On the contrary, finding a P-maximizer or a BRP-maximizer, if exist, may be an easier task since the literature provides simple characterizations of exact potentials and best-response potentials.¹⁶ In order to construct a nested best-response potential, we can use these characterizations. And, we can apply the conditions in terms of P-maximizers, BRP-maximizers, and nested BRP-maximizers regardless of an order and a partition of action sets.

¹⁶For example, Slade (1994), Monderer and Shapley (1996), Ui (2000), Uno (2007), Hino (2009), and Sandholm (2010) for exact potentials; Morris and Ui (2004) for best-response potentials.

5.3 Iterated LP-maximizer versus nested BRP-maximizer

Oyama and Tercieux (2009) introduce the iterated LP-maximizer as a specific and tractable form of iterated MP-maximizer.

Definition 5.12 (Morris and Ui, 2005; Oyama and Tercieux, 2009) An interval X^* of A is an *LP-maximizer set* of \mathbf{g}^N if there exist a function $v : A \rightarrow \mathbb{R}$ such that $X^* = \arg \max_{a \in A} v(a)$ and, for each $i \in N$, each $a_i \in A_i$ and any $\lambda_i \in \Delta(A_{-i})$, $a_i < \min X_i^*$ and

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i + 1, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i, a_{-i})$$

implies

$$\max_{a'_i \in Z_i^+} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a'_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}),$$

where $Z_i^+ = \{a_i + 1\}$ if $a_i + 1 < \min X_i^*$ and $Z_i^+ = X^*$ if $a_i + 1 = \min X_i^*$; and $a_i > \max X_i^*$ and

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i - 1, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i, a_{-i})$$

implies

$$\max_{a'_i \in Z_i^-} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a'_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}),$$

where $Z_i^- = \{a_i - 1\}$ if $a_i - 1 < \min X_i^*$ and $Z_i^- = X^*$ if $a_i - 1 = \min X_i^*$. Such a function v is called a *local potential*. If X^* is singleton $\{a^*\}$ then a^* is called an *LP-maximizer*. If the above weak inequalities are replaced with strict ones, X^* is called a *strict LP-maximizer set*, and v is called a *strict local potential*.

Definition 5.13 (Oyama and Tercieux, 2009) An action profile $a^* \in A$ is an *iterated strict LP-maximizer* of \mathbf{g}^N if there exists a sequence of intervals $A = X^0 \supset X^1 \supset \dots \supset X^K = \{a^*\}$ such that X^k is a strict LP-maximizer set of $\mathbf{g}^N|_{X^{k-1}}$ for each $k = 1, \dots, K$.

A game \mathbf{g}^N is said to have *diminishing marginal returns* for $i \in N$ if, for $a_i \in A_i \setminus \{0, m_i\}$ and $a_{-i} \in A_{-i}$, $g_i(a) - g_i(a_i - 1, a_{-i}) \geq g_i(a_i + 1, a_{-i}) - g_i(a)$.

Proposition 5.14 (Oyama and Tercieux, 2009) *If a^* is an iterated strict LP-maximizer of \mathbf{g}^N with associated intervals $(X^k)_{k=0}^K$ and strict local potentials $(v^k)_{k=0}^K$, and if, for each $k = 0, \dots, K$, $\mathbf{g}^N|_{X_i^{k-1} \times A_{-i}}$ or $v^k|_{X_i^{k-1} \times A_{-i}}$ has diminishing marginal returns for each $i \in N$ then a^* is an iterated strict MP-maximizer of \mathbf{g}^N with monotone potentials $(v^k)_{k=0}^K$.*

Corollary 5.15 *Suppose that \mathbf{g}^N has an iterated strict LP-maximizer a^* with associated intervals $(X^k)_{k=0}^K$ and strict local potentials $(v^k)_{k=0}^K$. For each $k = 0, \dots, K$, if $\mathbf{g}^N|_{X_i^{k-1} \times A_{-i}}$ or $v^k|_{X_i^{k-1} \times A_{-i}}$ is supermodular for each $i \in N$, and if $\mathbf{g}^N|_{X_i^{k-1} \times A_{-i}}$ or $v^k|_{X_i^{k-1} \times A_{-i}}$ has diminishing marginal returns for each $i \in N$ then a^* is robust to all elaborations in \mathbf{g}^N .*

Our condition does not imply the condition in terms of iterated LP-maximizers. To demonstrate it, we use the following characterization of LP-maximizers provided by Morris and Ui (2005).

Lemma 5.16 (Morris and Ui, 2005) *An action profile a^* is an LP-maximizer of \mathbf{g}^N if, and only if, there exist a function $v : A \rightarrow \mathbb{R}$ and a collection $(w_i(a_i))_{a_i \in A_i}$ of nonnegative numbers for $i \in N$ such that $X^* = \arg \max_{a \in A} v(a)$ and, for each $i \in N$ and each $a \in A$, $a_i < \min X_i^*$ implies*

$$w_i(a_i)[v(a) - v(a_i + 1, a_{-i})] \geq g_i(a) - g_i(a_i + 1, a_{-i}); \text{ and}$$

$a_i > \max X_i^*$ implies

$$w_i(a_i)[v(a) - v(a_i - 1, a_{-i})] \geq g_i(a) - g_i(a_i - 1, a_{-i}).$$

Example 5.17 Consider the game represented as in Table 1 again. Corollary 5.15 does not apply to the game. For any order on A , note that $\mathbf{g}^{\{1,2,3\}}$ is not supermodular for $i \in N$ and does not have diminishing marginal returns for $i \in N$. By Lemma 5.7 and Proposition 5.14, only A , $\{U\} \times \{L\} \times \{T\}$, $\{U\} \times \{L\} \times \{T, B_1\}$, $\{U\} \times \{L\} \times \{T, B_2\}$, or $\{U\} \times \{L\} \times \{T, B_1, B_2\}$

can be LP-maximizer sets. Clearly, A is an LP-maximizer set of \mathbf{g}^N such that an associated local potential is supermodular for $i \in N$ and has diminishing marginal returns for $i \in N$. We show that A is such a unique LP-maximizer set of \mathbf{g}^N . To show by contradiction, suppose firstly that $\{U\} \times \{L\} \times \{T\}$ is an LP-maximizer set such that an associated local potential v is supermodular for $i \in N$ and has diminishing marginal returns for $i \in N$. For $i \in N$, let $(w_i(a_i))_{a_i \in A_i}$ be an associated collection of nonnegative numbers. Fix any order on A_3 . We consider cases of orders on $A_1 \times A_2$ as in Table 5.

	$L < C < R$	$L < R < C$	$C < L < R$	$C < R < L$	$R < L < C$	$R < C < L$
$U < M < D$	case 1	case 1	case 1	case 1	case 1	case 1
$U < D < M$		case 2	case 3	case 4	case 4	case 1'
$M < U < D$			case 5	case 6	case 7	case 1'
$M < D < U$				case 2	case 3	case 1'
$D < U < M$					case 5	case 1'
$D < M < U$						case 1'

Table 5: cases of orders on A

Case 1 Since $\{U\} \times \{L\} \times \{T\}$ is an LP-maximizer set we have $0 > v(M, L, T) - v(U, L, T)$. Since v has diminishing marginal returns we have $v(M, L, T) - v(U, L, T) \geq v(D, L, T) - v(M, L, T)$. Since v is a local potential we have $w_1(D)[v(D, L, T) - v(M, L, T)] \geq g_1(D, L, T) - g_1(M, L, T) = 3$. This implies $w_1(D) > 0$ and $v(D, L, T) - v(M, L, T) \geq 3/w_1(D)$. These inequalities implies $0 > 0$, a contradiction.

Case 1' Since $\{U\} \times \{L\} \times \{T\}$ is an LP-maximizer set we have $0 > v(U, C, T) - v(U, L, T)$. Since v has diminishing marginal returns we have $v(U, C, T) - v(U, L, T) \geq v(U, R, T) - v(U, C, T)$. Since v is a local potential we have $w_2(R)[v(U, R, T) - v(U, C, T)] \geq g_1(U, R, T) - g_1(U, C, T) = 3$. This implies $w_2(R) > 0$ and $v(D, L, T) - v(M, L, T) \geq 3/w_2(R) > 0$. Thus, we have $0 > 0$, a contradiction.

Case 2 Since v is a local potential we have $w_1(D)[v(D, C, T) - v(U, C, T)] \geq g_1(D, C, T) - g_1(U, C, T) = 4$, $w_1(M)[v(M, R, T) - v(D, R, T)] \geq g_1(M, R, T) - g_1(D, R, T) = 4$, $w_2(R)[v(M, R, T) - v(M, L, T)] \geq g_2(M, R, T) - g_2(M, L, T) = 4$, and $w_2(C)[v(D, R, T) - v(D, C, T)] \geq g_2(D, R, T) - g_2(D, C, T) = 4$. These inequalities imply that $w_1(D), w_1(M), w_2(R), w_2(C) > 0$, and $v(D, C, T) - v(U, C, T) \geq 4/w_1(D)$, $v(M, R, T) - v(D, R, T) \geq 4/w_1(M)$, $v(M, R, T) - v(M, L, T) \geq 4/w_2(R)$,

and $v(D, R, T) - v(D, C, T) \geq 4/w_2(C)$. Moreover, we have $v(D, L, T) - v(U, L, T) \geq -2/w_1(D)$, $v(M, L, T) - v(D, L, T) \geq -3/w_1(M)$, $v(U, R, T) - v(U, L, T) \geq -2/w_2(R)$, and $v(U, C, T) - v(U, R, T) \geq -3/w_2(C)$, since v is a local potential and $w_1(D), w_1(M), w_2(R), w_2(C) > 0$. By summing up these inequalities, we have $2[v(D, C, T) + v(M, R, T) - v(D, R, T) - v(M, C, T)] \geq 2/w_1(D) + 1/w_1(M) + 2/w_2(R) + 1/w_1(C) > 0$. Since v is supermodular we have $0 \geq v(D, C, T) + v(M, R, T) - v(D, R, T) - v(M, C, T)$. Thus we have $0 > 0$, a contradiction.

Case 3 Since v is supermodular we have $v(D, L, T) - v(U, L, T) \geq v(U, L, T) - v(U, C, T)$. Since v is a local potential we have $w_1(D) > 0$ and $v(D, C, T) - v(U, C, T) \geq 4/w_1(D)$. Then we have $v(D, L, T) - v(U, L, T) \geq 4/w_1(D)$. Since v has diminishing marginal returns we have $v(D, R, T) - v(U, R, T) \geq v(M, R, T) - v(D, R, T)$. Since v is a local potential we have $w_1(M) > 0$ and $v(M, R, T) - v(D, R, T) \geq 4/w_1(M)$. Then we have $v(D, R, T) - v(U, R, T) \geq 4/w_1(M)$. Moreover, we have $w_2(R) > 0$, $v(M, L, T) - v(D, L, T) \geq -3/w_1(M)$, $v(M, R, T) - v(M, L, T) \geq 4/w_2(R)$, $v(U, R, T) - v(U, L, T) \geq -3/w_2(R)$, $v(M, R, T) - v(D, R, T) \geq 4/w_1(M)$ since v is a local potential. Summing up these inequalities, $v(D, L, T) - v(U, L, T) \geq 4/w_1(D)$ and $v(D, R, T) - v(U, R, T) \geq 4/w_1(M)$, we have $2[v(M, R, T) - v(U, L, T)] \geq 4/w_1(D) + 5/w_1(M) + 2/w_2(R) > 0$. Since $\{U\} \times \{L\} \times \{T\}$ is an LP-maximizer set we have $0 > v(M, R, T) - v(U, L, T)$. Thus, we have $0 > 0$, a contradiction.

Case 4 Since $\{U\} \times \{L\} \times \{T\}$ is an LP-maximizer set we have $0 > v(U, R, T) - v(U, L, T)$. Since v is supermodular we have $v(U, R, T) - v(U, L, T) \geq v(M, R, T) - v(M, L, T)$. Since v is a local potential we have $w_2(R) > 0$ and $v(M, R, T) - v(M, L, T) \geq 4/w_2(R) > 0$. Thus, we have $0 > 0$, a contradiction.

Case 5 Since $\{U\} \times \{L\} \times \{T\}$ is an LP-maximizer set we have $0 > v(D, L, T) - v(U, L, T)$. Since v is supermodular we have $v(D, L, T) - v(U, L, T) \geq v(D, C, T) - v(U, C, T)$. Since v is a local potential we have $w_1(D) > 0$ and $v(D, C, T) - v(U, C, T) \geq 4/w_1(D) > 0$. Thus, we have $0 > 0$, a contradiction.

Case 6 Since $\{U\} \times \{L\} \times \{T\}$ is an LP-maximizer set we have $0 > v(U, R, T) - v(U, L, T)$. Since v has diminishing marginal returns we have $v(U, R, T) - v(U, L, T) \geq v(U, C, T) - v(U, R, T)$. Since v is supermodular we have $v(U, C, T) - v(U, R, T) \geq v(D, C, T) - v(D, R, T)$. Since v is a local potential we have $w_1(D) > 0$ and $v(D, C, T) - v(D, R, T) \geq 4/w_1(D) > 0$. Thus, we have

$0 > 0$, a contradiction.

Case 7 Since v is a local potential we have $w_1(M) \geq 0$ and $w_1(M)[v(M, R, T) - v(U, R, T)] \geq 1$. This implies $w_1(M) > 0$. Since $w_1(M) > 0$ and v is a local potential, we have $v(M, C, T) - v(U, C, T) \geq 0$, or equivalently, $0 \geq v(U, C, T) - v(M, C, T)$. Since v has diminishing marginal returns we have $v(U, C, T) - v(M, C, T) \geq v(D, C, T) - v(U, C, T)$. Since v is a local potential we have $w_1(D) > 0$ and $v(D, C, T) - v(U, C, T) \geq 4/w_1(D) > 0$. Thus, we have $0 > 0$, a contradiction.

Other cases Since players 1 and 2 have symmetric payoffs, we can apply the above arguments to the other cases. Hence, $\mathbf{g}^{\{1,2,3\}}$ has no LP-maximizer set $\{U\} \times \{L\} \times \{T\}$ such that an associated local potential v is supermodular for $i \in N$ and has diminishing marginal returns for $i \in N$.

In the above arguments, we use only information on payoffs of players 1 and 2. So, we can apply the same arguments to show that $\mathbf{g}^{\{1,2,3\}}$ has no LP-maximizer set $\{U\} \times \{L\} \times \{T, B_1\}$, $\{U\} \times \{L\} \times \{T, B_2\}$, or $\{U\} \times \{L\} \times \{T, B_1, B_2\}$ such that an associated local potential v is supermodular for $i \in N$ and has diminishing marginal returns for $i \in N$. Therefore, A is a unique LP-maximizer set of \mathbf{g}^N such that an associated local potential is supermodular for $i \in N$ and has diminishing marginal returns for $i \in N$. Hence, Corollary 5.15 does not apply to the game.

Remark 5.18 In fact, Morris and Ui (2005) define a more general form of local potentials, a \mathcal{P} -measurable local potential for a partition \mathcal{P} over the set A of action profiles, and provide a sufficient condition in terms of \mathcal{P} -measurable LP-maximizers. It is not sure whether or not our condition implies the condition in terms of \mathcal{P} -measurable LP-maximizers. Consider the game in Table 1 again. Assume an order on A such that $U < D < M$, $L < R < C$, and $T < B_1 < B_2$. Let $\mathcal{P} = \{\{U\}, \{D, M\}\} \times \{\{L\}, \{R\}, \{C\}\} \times \{\{T\}, \{B_1\}, \{B_2\}\}$. We can find an LP-maximizer (U, L, T) such that an associated \mathcal{P} -measurable local potential is supermodular for $i \in N$ and has diminishing marginal returns for $i \in N$. Such a local potential v is given by Table 6. Thus, the condition in terms of \mathcal{P} -measurable LP-maximizers by Morris and Ui (2005) applies to the game.

On the other hand, since a simple characterization for \mathcal{P} -measurable LP-maximizers is un-

T	L	R	C	B_1	L	R	C	B_2	L	R	C
U	5	2	-43	U	4	1	-40	U	4	1	-39
D	-16	-1	0	D	-13	2	3	D	-12	3	4
M	-16	-1	0	M	-13	2	3	M	-12	3	4

Table 6: A \mathcal{P} -measurable local potential v with supermodular and diminishing marginal returns known, finding a \mathcal{P} -measurable LP-maximizers is a hard task, as we pointed out in Remark 5.11.

5.4 The other conditions

Our condition does not imply the other sufficient conditions in the literature. Kajii and Morris (1997a) show that a unique correlated equilibrium is robust to all elaborations. Kajii and Morris (1997a) also show that a \mathbf{p} -dominant equilibrium with low \mathbf{p} is robust to all elaborations. Tercieux (2006) shows that a unique correlated equilibrium whose support is \mathbf{p} -best-response set with low \mathbf{p} introduced by Tercieux (2004) is robust to all elaborations. The condition in terms of \mathbf{p} -best-response sets unifies two conditions of Kajii and Morris (1997a). Oyama and Tercieux (2009) introduce the iterated strict \mathbf{p} -best-response equilibrium and show that an iterated strict \mathbf{p} -best-response equilibrium with low \mathbf{p} is robust to all elaborations.

Our condition does not imply the above conditions. Indeed, our condition applies to the game in Table 1 as shown in Example 5.1. However, by Example 5.17, the conditions in terms of unique correlated equilibria, \mathbf{p} -dominant equilibria, \mathbf{p} -best-response sets, and iterated strict \mathbf{p} -best-response equilibria, does not apply to the game since the above conditions are special cases of condition in terms of iterated LP-maximizers (Corollary 5.15).¹⁷

Remark 5.19 The conditions in terms of unique correlated equilibria and \mathbf{p} -dominant equilibria, LP-maximizers, MP-maximizers, \mathbf{p} -best-response sets, iterated strict \mathbf{p} -best-response equilibria, iterated LP-maximizers, and iterated MP-maximizers does not imply our condition. Indeed, in these conditions note that the conditions in terms of unique correlated equilibria and

¹⁷Note that $\mathbf{g}^{\{1,2,3\}}$ has multiple correlated equilibria $\mu \in \Delta(A)$ such that $\mu(U, L, T) = 1$ and $\mu' \in \Delta(A)$ such that $\mu'(M, C, T) = 3/14$, $\mu'(M, R, T) = \mu'(D, C, T) = 3/28$, $\mu'(M, C, B_1) = 2/7$, and $\mu'(M, R, B_1) = \mu'(D, C, B_1) = 1/7$.

in terms of **p**-dominant equilibria are the strongest. The condition in terms of unique correlated equilibria applies to matching pennies games but our condition does not apply to it, since the game has a best-response cycle. We can also show that the conditions in terms of **p**-dominant equilibria, as well as the conditions in terms of LP-maximizers, MP-maximizers, **p**-best-response sets, iterated strict **p**-best-response equilibria, iterated LP-maximizers, and iterated MP-maximizers, apply to the game in Table 7 but our condition does not apply to it. Indeed, the game has no nested BRP-maximizer since it has a strict best-response cycle $(M, C) \rightarrow (D, C) \rightarrow (D, R) \rightarrow (M, R) \rightarrow (M, C)$, and for two-person games, a best-response potential is equivalent to a nested best-response potential. On the other hands, we can show that (U, L) is **p**-dominant equilibrium for $p_1, p_2 > 1/6$.

	L	C	R
U	5, 5	0, 0	0, 0
M	0, 0	0, 1	1, 0
D	0, 0	1, 0	0, 1

Table 7: (g_1, g_2) has no nested BRP-maximizer

5.5 Iterative construction versus nested construction

At a general level the “nested construction” by Uno (2007) is related to the “iterative construction” by Oyama and Tercieux (2009). Both constructions are defined by applying a concept in the literature iteratively. To compare between these constructions, we apply the iterative construction to BRP-maximizer. To do this, we define the BRP-maximizer sets.

Definition 5.20 For $i \in N$, let $X_i^* \subseteq A_i$, and let $X^* := \prod_{i \in N} X_i^*$. X^* is a *BRP-maximizer set* of \mathbf{g}^N if there exists a function $f : A \rightarrow \mathbf{R}$ such that $X^* = \arg \max_{a \in A} f(a)$ and, for each $i \in N$ and all $\lambda_i \in \Delta(A_{-i})$, $X_i \subseteq BR_i^f(\lambda_i)$ implies $X_i \cap BR_i^{g_i}(\lambda_i) \neq \emptyset$.

Definition 5.21 An action profile a^* is said to be an *iterated BRP-maximizer* if there exists a sequence of subsets of action profile $A = X^0 \supset X^1 \supset \dots \supset X^K = \{a^*\}$ such that, for each $k = 1, \dots, K$, X^k is a BRP-maximizer set in the game restricted X^{k-1} .

T	L	R	B	L	R
U	3, 3, 3	0, 0, 0	U	0, 0, 0	2, 2, 2
D	0, 0, 1	1, 1, 0	D	1, 1, 0	0, 0, 1

Table 8: A game (g_1, g_2, g_3)

Remark 5.22 It is not sure that an (iterated) BRP-maximizer of Definitions 5.20 and 5.21 is robust to canonical elaborations since we cannot apply the proofs by Morris and Ui (2005) and Oyama and Tercieux (2009) directly.

The condition in terms of iterated BRP-maximizers neither implies nor is implied by the condition in terms of nested BRP-maximizers.

Example 5.23 Consider the game $\mathbf{g}^{\{1,2,3\}}$ represented as Table 8. The game has a unique BRP-maximizer set A . Indeed, note that if X^* is a BRP-maximizer then it is an MP-maximizer. By Lemma 5.7 only $\{(U, L, T)\}$, $\{(U, R, B)\}$, or A may be a BRP-maximizer. Suppose that f is a best-response potential with a BRP-maximizer set $\{(U, L, T)\}$. Let $\lambda_2 \in \Delta(A_{-2})$ be such that $\lambda_2(D, T) = 1$, let $\lambda'_2 \in \Delta(A_{-2})$ be such that $\lambda'_2(D, B) = 1$, let $\lambda_3 \in \Delta(A_{-3})$ be such that $\lambda_3(D, L) = 1$, and let $\lambda'_3 \in \Delta(A_{-3})$ be such that $\lambda'_3(D, R) = 1$. Then we have $BR_2^{g_2}(\lambda_2) = \{R\}$, $BR_2^{g_2}(\lambda'_2) = \{L\}$, $BR_3^{g_3}(\lambda_3) = \{T\}$, and $BR_3^{g_3}(\lambda'_3) = \{B\}$. Since f is a best-response potential, we have $f(D, L, T) = f(D, R, T) = f(D, L, B) = f(D, R, B)$. And, since $\{(U, L, T)\}$ is a BRP-maximizer set, we have $f(U, L, T) > f(U, R, T)$. Let $\lambda''_2 \in \Delta(A_{-2})$ be such that $\lambda''_2(U, T) = 1/6$ and $\lambda''_2(D, T) = 5/6$. Since $[f(U, L, T) + 5f(D, L, T)]/6 > [f(U, R, T) + 5f(D, R, T)]/6$ we have $BR_2^f(\lambda''_2) = \{L\}$. On the other hand, we have $BR_2^{g_2}(\lambda''_2) = \{R\}$. Since f is a best-response potential we must have $\{R\} \cap BR_2^f(\lambda''_2) \neq \emptyset$, a contradiction. Thus, $\{(U, L, T)\}$ is not a BRP-maximizer set. By the similar arguments, we can show that $\{(U, R, B)\}$ is also not a BRP-maximizer set. Thus, the game has a unique BRP-maximizer set A . This implies that there is no iterative BRP-maximizer in the game.

However, $\mathbf{g}^{\{1,2,3\}}$ has a nested best response potential. Indeed, $(f_{\{3\}}^1, f_{\{1,2\}}^1)$ represented in Table 9 is a $\{\{3\}, \{1, 2\}\}$ -best response potential of $\mathbf{g}^{\{1,2,3\}}$, where $f_{\{3\}}^1(\cdot) = g_3(\cdot)$ and $f_{\{1,2\}}^1(\cdot) = g_1(\cdot) = g_2(\cdot)$, and then considering the $\{\{3\}, \{1, 2\}\}$ -best response potential $(f_{\{3\}}^1, f_{\{1,2\}}^1)$ as a two-person game, we can show that $\mathbf{f}^{\{1,2,3\}} = (f)$ represented in Table 10 is a $\{\{1, 2, 3\}\}$ -best

	U, L	U, R	D, L	D, R
T	3, 3	0, 0	1, 0	0, 1
B	0, 0	2, 2	0, 1	1, 0

Table 9: A partition $\{\{3\}, \{1, 2\}\}$ -potential $(f_{\{3\}}^1, f_{\{1,2\}}^1)$

	U, L	U, R	D, L	D, R
T	3	0	1	0
B	0	2	0	1

Table 10: A nested potential f

response potential of $(f_{\{3\}}^1, f_{\{1,2\}}^1)$. Thus $\mathbf{g}^{\{1,2,3\}}$ has a nested best response potential f .

Example 5.24 Consider the game in Example 5.19 again. The game has an (iterated) BRP-maximizer (U, L) of Definitions 5.20 and 5.21 such that an (iterated) best-response potential is represented as in Table 11. But it has no (nested) BRP-maximizer of Definitions 3.1 and 3.3, which is shown in Example 5.19.

	L	C	R
U	5	0	0
M	0	1	1
D	0	1	1

Table 11: An (iterated) best-response potential

A Appendix

A.1 \mathcal{P} -mesurable BRP-maximizer versus nested BRP-maximizer

Morris and Ui (2005) introduce a generalized version of best-response potential. Let $\mathcal{P}_i \subseteq 2^{A_i} \setminus \emptyset$ be a partition of A_i . We write $\mathcal{P} = \{\prod_{i \in N} X_i | X_i \in \mathcal{P}_i \text{ for } i \in N\}$. A function $v : A \rightarrow \mathbb{R}$ is \mathcal{P} -mesurable if, for $X \in \mathcal{P}$ and for $a, a' \in X$, $v(a) = v(a')$.

Definition A.1 (Morris and Ui, 2005) A \mathcal{P} -mesurable function $v : A \rightarrow \mathbb{R}$ is a *best-response potential* of \mathbf{g}^N if, for each $i \in N$, $X_i \cap BR_i^{g_i}(\lambda_i) \neq \emptyset$ for all $X_i \in \mathcal{P}_i$ and $\lambda_i \in \Delta(A_{-i})$ such that $X_i \subseteq BR_i^v(\lambda_i)$. A partition element $X^* \in \mathcal{P}$ is a BRP-maximizer if $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \in X \setminus X^*$.

Remark A.2 If $\mathcal{P} = \prod_{i \in N} \{a_i | a_i \in A_i\}$, the \mathcal{P} -measurable best-response potentials are given by Definition 3.1 as mentioned in Morris and Ui (2005).

It is clear that a \mathcal{P} -measurable function $v : A \rightarrow \mathbb{R}$ is a best-response potential of \mathbf{g}^N if and only if, for each $i \in N$, $X_i \in \mathcal{P}_i$, and $\lambda_i \in \Delta(A_{-i})$, $X_i \cap BR_i^{g_i}(\lambda_i) = \emptyset$ implies that $X_i \not\subseteq BR_i^v(\lambda_i)$.

For $i \in N$ and a function $f : A \rightarrow \mathbb{R}$, let denote $BR_i^f(a_{-i}) := \arg \max_{a_i \in A_i} f(a)$ for $a_{-i} \in A_{-i}$ by abuse of notation. We provide a necessary condition for existence of \mathcal{P} -measurable-best-response potential.

Lemma A.3 Assume that for each $i \in N$ and each $a_{-i} \in A_{-i}$, there exists $a_i \in A_i$ such that $\{a_i\} = BR_i^{g_i}(a_{-i})$. If \mathbf{g}^N has a \mathcal{P} -measurable best-response potential then, for each $i \in N$, for each $X_{-i} \in \mathcal{P}_{-i}$, for each $a_{-i}, a'_{-i} \in X_{-i}$, there exists a $X_i \in \mathcal{P}_i$ such that $X_i \cap BR_i^{g_i}(a_{-i}) \neq \emptyset$ and $X_i \cap BR_i^{g_i}(a'_{-i}) \neq \emptyset$.

Proof. Suppose that there is $i \in N$, $X_{-i} \in \mathcal{P}_{-i}$ and $a_{-i}, a'_{-i} \in X_{-i}$ such that, for each $X_i \in \mathcal{P}_i$, $X_i \cap BR_i^{g_i}(a_{-i}) = \emptyset$ or $X_i \cap BR_i^{g_i}(a'_{-i}) = \emptyset$. Assume that \mathbf{g}^N has a \mathcal{P} -measurable best-response potential v . Let $\lambda_i, \lambda'_i \in \Delta(A_{-i})$ be such that $\lambda_i(a_{-i}) = 1$ and $\lambda_i(a'_{-i}) = 1$. Since i 's best responses against a_{-i} and a'_{-i} is singleton respectively and for each $X_i \in \mathcal{P}_i$, $X_i \cap BR_i^{g_i}(a_{-i}) = \emptyset$ or $X_i \cap BR_i^{g_i}(a'_{-i}) = \emptyset$, we have $X_i, X'_i \in \mathcal{P}$ such that $X_i \neq X'_i$, $X_i \cap BR_i^{g_i}(\lambda_i) \neq \emptyset$ and $X'_i \cap BR_i^{g_i}(\lambda'_i) \neq \emptyset$. Since v is a \mathcal{P} -measurable best-response potential and i 's best responses against a_{-i} and a'_{-i} is singleton, for each $X''_i \in \mathcal{P}_i \setminus X_i$, $X''_i \not\subseteq BR_i^v(a_{-i})$; and for each $X''_i \in \mathcal{P}_i \setminus X'_i$, $X''_i \not\subseteq BR_i^v(a'_{-i})$, or equivalently, $X_i = BR_i^v(a_{-i})$ and $X'_i = BR_i^v(a'_{-i})$. Since v is a \mathcal{P} -measurable, $v(a''_i, a_{-i}) = v(a''_i, a'_{-i})$ for every $a''_i \in A_{-i}$. Then we have $X_i = BR_i^v(\lambda_i) = BR_i^v(\lambda'_i) = X'_i$, which contradicts to $X_i \neq X'_i$. ■

Example A.4 Consider the game in Example 5.1 again. We can show that for only the partition $\mathcal{P} = \{A\}$ in 125 possible partitions the game has a \mathcal{P} -measurable best-response potential. Indeed, first, let consider $\mathcal{P} = \{\{U\}, \{M\}, \{D\}\} \times \{\{L\}, \{C\}, \{R\}\} \times \{\{T\}, \{B\}\}$. The \mathcal{P} -measurable best-response potentials is given by Definition 3.1 as mentioned in Morris and Ui (2005). By Example 5.1 there is no \mathcal{P} -measurable best-response potential.

Next consider $\mathcal{P} = \{\{U\}, \{M\}, \{D\}\} \times \{\{L\}, \{C\}, \{R\}\} \times \{\{T, B_1, B_2\}\}$. Let $i = 1$, $X_{-i} = \{C\} \times \{T, B_1, B_2\}$, $a_{-i} = (C, T)$, $a'_{-i} = (C, B_1)$. Since $\{D\} = BR_i^{g_i}(a_{-i})$ and $\{M\} = BR_i^{g_i}(a'_{-i})$,

there does not exist a $X_i \in \mathcal{P}_i$ such that $X_i \cap \{D\} \neq \emptyset$ and $X_i \cap \{M\} \neq \emptyset$. By Lemma A.3, \mathbf{g}^N has no \mathcal{P} -measurable best-response potential. By the similar arguments, for the other partitions except the partition $\{A\}$, we can show that \mathbf{g}^N has no \mathcal{P} -measurable best-response potential. Thus, for only the partition $\mathcal{P} = \{A\}$ the game has a \mathcal{P} -measurable best-response potential.

References

- [1] Dubey, P., Haimanko, O., and Zapechelnyuk, A., (2006) “Strategic complements and substitutes, and potential games,” *Games and Economic Behavior* **54**, 77-94.
- [2] Carlsson, H., and van Damme, E., (1993) “Global games and equilibrium selection,” *Econometrica* **61**, 989-1018.
- [3] Hino, Y., (2009) “An improved algorithm for detecting potential games,” *International Journal of Game Theory* (forthcoming)
- [4] Kajii, A., and Morris, S., (1997a) “The robustness of equilibria to incomplete information,” *Econometrica* **65**, 1283-1309.
- [5] Kajii, A., and Morris, S., (1997b) “Refinements and higher order beliefs: a unified survey,” mimeo.
- [6] Kukushkin, N. S., (2004) “Best response dynamics in finite games with additive aggregation,” *Games and Economic Behavior* **48**, 94-110.
- [7] Monderer, D., (2007) “Multipotential games,” in *Twentieth International Joint Conference on Artificial Intelligence (IJCAI-07)*, 1422-1427.
- [8] Monderer, D., and Shapley, L., (1996) “Potential games,” *Games and Economic Behavior* **14**, 124-143.
- [9] Morris, S., and Ui, T., (2004) “Best response equivalence,” *Games and Economic Behavior* **49**, 260-287.

- [10] Morris, S., and Ui, T., (2005) “Generalized potentials and robust sets of equilibria,” *Journal of Economic Theory* **124**, 45-78.
- [11] Oyama, D., and Tercieux, O., (2009) “Iterated potential and robustness of equilibria,” *Journal of Economic Theory* **144** (4), 1726-1769.
- [12] Rubinstein, A., (1989) “The electronic mail game: strategic behavior under almost common knowledge,” *The American Economic Review* **79**, 385-391.
- [13] Sandholm, W., (2010) “Decompositions and potentials for normal form games,” *Games and Economic Behavior* **70** (2), 446-456.
- [14] Tercieux, O., (2004) “p-Best response sets,” *Journal of Economic Theory* **131**, 45-70.
- [15] Tercieux, O., (2006) “p-Best response sets and the robustness of equilibria to incomplete information,” *Games and Economic Behavior* **56**, 371-384.
- [16] Ui, T., (2000) “A Shapley value representation of potential games,” *Games and Economic Behavior* **31**, 121-135.
- [17] Ui, T., (2001) “Robust equilibria of potential games,” *Econometrica* **69**, 1373-1380.
- [18] Uno, H., (2007) “Nested potential games,” *Economics Bulletin* **3** (17), 1-8
- [19] Voorneveld, M., (2000) “Best-response potential games,” *Economics Letters* **66**, 289-295.

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